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EMBEDDING OF A PSEUDO-POINT RESIDUAL DESIGN INTO A MÖBIUS PLANE*

Agnes Hui CHAN

Department of Mathematics, Northeastern University, Boston, MA 02115, USA

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Let $\mathfrak{A}$ be a class of subsets of a finite set X. Elements of $\mathfrak{A}$ are called *blocks*. Let v, t, λ and k be nonnegative integers such that $v \geq k \geq t \geq 0$. A pair $(X, \mathfrak{A})$ is called a (v, k, λ) *t-design*, denoted by $S_\lambda(t, k, v)$, if (1) $|X| = v$, (2) every t-subset of X is contained in exactly λ blocks and (3) for every block A in $\mathfrak{A}$, $|A| = k$. A Möbius plane M is an $S_1(3, q+1, q^2+1)$ where q is a positive integer. Let ∞ be a fixed point in M. If ∞ is deleted from M, together with all the blocks containing ∞, then we obtain a point-residual design M^*. It can be easily checked that M^* is an $S_q(2, q+1, q^2)$. Any $S_q(2, q+1, q^2)$ is called a pseudo-point-residual design of order q, abbreviated by PPRD(q). Let A and B be two blocks in a PPRD(q)M^*. A and B are said to be tangent to each other at z if and only if $A \cap B = \{z\}$. M^* is said to have the *Tangency Property* if for any block A in M^*, and points x and y such that $x \in A$ and $y \notin A$, there exists at most one block containing y and tangent to A at x. This paper proves that any PPRD(q)M^* is uniquely embeddable into a Möbius plane if and only if M^* satisfies the Tangency Property.

1. Introduction

Let $(X, \mathfrak{A})$ be an ordered pair, where X is a finite set and $\mathfrak{A}$ is a collection of subsets of X. Members of X are called *points* and elements of $\mathfrak{A}$ are called *blocks*. Let N_0 denote the set of nonnegative integers, and $v, k, \lambda, t \in N_0$ such that $v \geq k \geq t \geq 0$. A structure $D = (X, \mathfrak{A})$ is called a *t-design*, denoted by $S_\lambda(t, k, v)$, if and only if (1) every t-subset of X is contained in exactly λ blocks, (2) for every block A is $\mathfrak{A}$, $|A| = k$ and (3) $|X| = v$.

Let ∞ be a fixed point in D. A *point-residual* design of D with respect to ∞ is a design $D^* = (X^*, \mathfrak{A}^*)$, where $X^* = X - \{\infty\}$, and $\mathfrak{A}^* = \mathfrak{A}$ — {blocks containing ∞}. If D is an $S_1(t, k, v)$, then it can be easily seen that D^* is an $S_\lambda(t-1, k, v-1)$. A design with parameters as those of a point-residual design is called a *pseudo-point-residual* design. A pseudo-point-residual design D^* is said to be *embeddable* if and only if there exists a design D whose point-residual design is isomorphic to D^*.

There have been many investigations into the embedding problems of various pseudo-residual designs. Connor and Hall [5] proved the embedding theorem for

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a pseudo-block-residual design of an $S_2(2, k, v)$. Bose, Singhi and Shrikhande [2] extended the result to $S_\lambda(2, k, v)$ for $\lambda \geq 3$. Chan and Ray–Chaudhuri [4] proved the embedding theorem for a pseudo-block-residual design of a Möbius plane. In this paper we consider the dual of the Möbius plane problem and show the embedding of a pseudo-point-residual design of a Möbius plane. In the last section of this paper, we exhibit a pseudo-point-residual design of the Möbius plane, which does not satisfy the Tangency Condition and is not embeddable.

2. Motivation and statement of theorem

A Möbius plane M is an $S_1(3, q+1, q^2+1)$, where q is a positive integer. If M' is a point-residual design obtained from M, then M' is an $S_q(2, q+1, q^2)$. Any 2-design with the same parameters is called a pseudo-point-residual design of order q, abbreviated by PPRD(q). Our purpose is to reconstruct the Möbius plane from a given PPRD(q). Let us first study properties possessed by a Möbius plane M. A block A is said to be *tangent* to another block A' at a point x if and only if $A \cap A' = \{x\}$. A and A' are said to be *secant* to one another if $|A \cap A'| \geq 2$. Let A be a block in M containing a point x, and y is a point not in A. Then there exists at most one block containing y which is tangent to A at x. We shall show that this is a sufficient condition for a PPRD(q) to be embeddable.

Definition. Let A be a block in a design and x be a point in A. For every point y not in A, the *Tangency Condition* requires that there exists at most one block containing y and tangent to A at x.

Theorem. *Let D^* be an $S_q(2, q+1, q^2)$. D^* is uniquely embeddable in a Möbius plane M if and only if D^* satisfies the Tangency Condition.*

3. Proof of theorem

Let D^* be an $S_q(2, q+1, q^2)$ and A be a block in D^*. If x is a point in D^*, it can be easily checked that x is contained in q^2-1 blocks. Furthermore, there are $q^2(q-1)$ blocks in D^*.

Lemma 1. *If x is a point in A, then there exists at least $q-2$ blocks that are tangent to A at x.*

Proof. For every point y in A, $y \neq x$, there are $q-1$ blocks distinct from A, that contain x and y. Hence there are at most $q(q-1)$ blocks containing x and secant to A. But every point in D^* is contained in $(q+1)(q-1)$ blocks. Hence there are at least $q-2$ blocks tangent to A at x.

Lemma 2. *Let D^* be a* PPRD(q) *such that D^* satisfies the Tangency Condition. If A is a block in D^* and x is a point in A, then there exists exactly $q-2$ blocks tangent to A at x.*

Proof. Since for every point y not in A, there exists at most one block tangent to A at x which contains y, the tangents of A at x are mutually tangent. Hence, if t denotes the number of tangents of A at x, then

$$qt+q+1 \leqslant q^2 \quad \text{and} \quad t \leqslant q-1-1/q.$$

Since t is an integer, $t \leqslant q-2$. From the previous lemma, there are exactly $q-2$ blocks tangent to A at x.

Henceforth, we shall assume that D^* satisfies the Tangency Condition.

Lemma 3. *Every 3 distinct points in D^* are contained in at most one block.*

Proof. Suppose x, y and z are distinct points contained in two blocks A and A'. Consider the blocks containing x and secant to A. Since A' is a secant of A which intersects A at both y and z, there are at most $q(q-1)-1$ secants of A that contain x. This implies that there are at least $q-1$ blocks tangent to A at x. But this contradicts the fact that there exist exactly $q-2$ such blocks. Hence every 3 distinct points are contained in at most one block.

We shall call a set of points to be *concyclic* if they are contained in a block.

Lemma 4. *For every two distinct points x and y in D^*, there are exactly $q-2$ points z in D^* such that x, y and z are not concyclic.*

Proof. Let $A_1, \ldots, A_q$ be the blocks containing x and y. Since no 3 points are contained in more than one block, $A_1, \ldots, A_q$ intersect each other at x and y only. Hence, there are $2+q(q-1)$ points that are covered by $A_1, \ldots, A_q$. But $v=q^2$, this implies that there are exactly $q-2$ points z, such that x, y and z are not concyclic.

Lemma 5. *If (x, y, z) and (x, y, z') are two nonconcyclic triples, then (x, z, z') is a nonconcyclic triple.*

Proof. Suppose x, z and z' are concyclic, then there exists a unique block A containing them. Consider the point y, y is not in A. For every point v in A, $v \neq x, z, z'$, there exists at most one block containing x, y and v. Since there are $q-2$ points in A that are distinct from x, z and z', there are at most $q-2$ blocks containing x and y that are secant to A. But x and y are contained in q blocks, hence there are at least two blocks containing y which are tangent to A at x. This contradicts the Tangency Condition. Therefore, x, z and z' are nonconcyclic.

For every pair of distinct points x and y in D^*, let us define $A(x, y) = \{z \mid (x, y, z)$ is a nonconcyclic triple$\}$. From the above lemma, we observe:

Lemma 6. *If u and w are two distinct points in $A(x, y)$, then $A(x, y) = A(u, w)$.*

Proof. Let $z \in A(x, y)$. Since (x, y, z) and (x, y, u) are nonconcyclic triples, (x, z, u) is nonconcyclic. Similarly (x, z, w) is a nonconcyclic triple. But by the previous lemma, z, u and w are nonconcyclic. Hence, $z \in A(u, w)$. Thus $A(x, y) \subseteq A(u, w)$. By symmetry, $A(u, w) \subseteq A(x, y)$ and the proof is complete.

Proof of Main Theorem. Let $D^* = (x, \mathfrak{A})$ be an PPRD(q) such that D^* satisfies the Tangency Condition. We define

$$\bar{X} = X \cup \{\infty\} \quad \text{and} \quad \bar{\mathfrak{A}} = \mathfrak{A} \cup \{\bar{A}(x, y) \mid x \neq y, x, y \in X\}$$

where $\bar{A}(x, y) = A(x, y) \cup \{x, y, \infty\}$. We shall show that $\bar{D} = (\bar{X}, \bar{\mathfrak{A}})$ is a Möbius plane of order q. Clearly, $|\bar{X}| = q^2 + 1$, and every block in $\bar{\mathfrak{A}}$ contains $q + 1$ points. We are left to show that every 3 distinct points in $\bar{X}$ are contained in a unique block.

Let x, y and z be three distinct points in $\bar{X}$. We first show that there exists at least a block containing them. If $z = \infty$, then the block $\bar{A}(x, y)$ contains x, y and z. If x, y and z are in X, and (x, y, z) is a concyclic triple in X, then $\bar{A}(x, y)$ contains x, y and z. Hence, every 3 distinct points are contained in at least one block.

Next, we shall count the number of blocks in $\bar{\mathfrak{A}}$. Let us compute the number of triples $(x, y, \bar{A}(x, y))$ such that $x \neq y$. For every distinct pair (x, y), there exists a unique $\bar{A}(x, y)$. Hence, there are $q^2(q^2 - 1)$ such triples.

On the other hand, if $\bar{A}(x, y)$ is a block in $\bar{\mathfrak{A}}$, then there are $q(q-1)$ choices of (x, y). Therefore, number of blocks $\bar{A}(x, y) = q(q+1)$. But $|\mathfrak{A}| = q^2(q-1)$, hence $|\bar{\mathfrak{A}}| = q^3 + q$.

Next, it can be easily computed that the average number of blocks containing 3 distinct points, is one. As a result, every 3 distinct points are contained in a unique block.

Thus, $(\bar{X}, \bar{\mathfrak{A}})$ is an $S_1(3, q+1, q^2+1)$ or a Möbius plane.

4. An example

In this section we shall show a PPRD(3) which does not satisfy the Tangency Condition and is not embeddable. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Each block is of size 4, and there are 18 blocks. Let the blocks be

0123	0456	1568
0124	0478	2348
0157	1267	2357
0258	1358	2456
0368	1346	2678
0367	1478	3457.

It can be easily checked that this is an $S_3(2,4,9)$. Let us consider the blocks $A=(0,1,2,3)$ and $B=(0,4,5,6)$ and $C=(0,4,7,8)$. Both B and C contain the point 4 and tangent to A at 0; this violates the Tangency Condition, and it is obvious that this design cannot be embedded into the Möbius plane of order 3.

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